Paignton. 30 July 96.

Dear Fitzgerald,

Your integrals are particular forms of those occurring in an unpublished investigation of mine. I showed in paper "On the General solution of M's eqns" that

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\lambda m \sin\frac{n}{v}\sqrt{\lambda^2-r^2}}{(\lambda^2-r^2)^{\frac{1}{2}}} = I_0\left[\left(\frac{n^2}{v^2}-m^2\right)^{\frac{1}{2}} r\right] \text{ if } \frac{n}{v}>m$$
$$= 0 \text{ if } \frac{n}{v}<m.$$

Write $+r^2$ for $-r^2$ and we make

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\lambda m \sin\frac{n}{v}\sqrt{\lambda^2+r^2}}{(\lambda^2+r^2)^{\frac{1}{2}}}\,d\lambda = J_0\left[\left(\frac{n^2}{v^2}-m^2\right)^{\frac{1}{2}} r\right] \text{ if } \frac{n}{v}>m$$
$$= 0 \text{ if } \frac{n}{v}<m.$$

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In the unpublished investign referred to I get the more complete results

$$m>\frac{n}{v}\begin{cases} 0 = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin\frac{n}{v}(r^2+\lambda^2)^{\frac{1}{2}}\cos m\lambda}{(r^2+\lambda^2)^{\frac{1}{2}}}\,d\lambda \\ K_0\left[\left(m^2-\frac{n^2}{v^2}\right)^{\frac{1}{2}} r\right] = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos\frac{n}{v}(r^2+\lambda^2)^{\frac{1}{2}}\cos m\lambda}{(r^2+\lambda^2)^{\frac{1}{2}}}\,d\lambda \end{cases}$$

$$m<\frac{n}{v}\begin{cases} G_0\left[\left(\frac{n^2}{v^2}-m^2\right)^{\frac{1}{2}} r\right] = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos\frac{n}{v}(\lambda^2+r^2)^{\frac{1}{2}}\cos m\lambda}{(r^2+\lambda^2)^{\frac{1}{2}}}\,d\lambda \\ J_0\left[\left(\frac{n^2}{v^2}-m^2\right)^{\frac{1}{2}} r\right] = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin\frac{n}{v}(\lambda^2+r^2)^{\frac{1}{2}}\cos m\lambda}{(r^2+\lambda^2)^{\frac{1}{2}}}\,d\lambda \end{cases}$$

Take $m=0$ in the last two to get your cases.

Say
$$\left.\begin{aligned} J_0(sr) &= \frac{2}{\pi}\int_0^\infty \frac{\sin s(\lambda^2+r^2)^{\frac{1}{2}}}{(\lambda^2+r^2)^{\frac{1}{2}}}\,d\lambda \\ G_0(sr) &= \frac{2}{\pi}\int_0^\infty \frac{\cos s(\lambda^2+r^2)^{\frac{1}{2}}}{(\lambda^2+r^2)^{\frac{1}{2}}}\,d\lambda \end{aligned}\right\}$$

which are what you want I think.

I use my own symmetrical notation for the Bn fns of the second kind. E.g.

$$K_0(qr) = \left(\frac{2}{\pi q r}\right)^{\frac{1}{2}}\varepsilon^{-qr}\left\{1-\frac{1^2}{8qr}+\frac{1^2 3^2}{(8qr)^2}-\cdots\right\} = \frac{2}{\pi}\left\{\begin{aligned}&\frac{q^2r^2}{2^2}+\left(1+\frac{1}{2}\right)\frac{q^4r^4}{2^2 4^2}+\cdots \\ &- I_0(qr)\left\{\log\left(\tfrac{1}{2}qr\right)+.5772\right\}\end{aligned}\right\}$$

Put $q = si$, then $$K_0(qr) = G_0(sr) - i\,J_0(sr)$$

where $J_0(sr)$ is as usual $= 1 - \frac{s^2r^2}{2^2} + \frac{s^4r^4}{2^24^2}$

$$= \left(\frac{2}{\pi sr}\right)^{\frac{1}{2}}\left\{P\cos + Q\sin\right\}\left(sr - \frac{\pi}{4}\right)$$

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where $P = 1 - \frac{1^2 3^2}{\lfloor 2\,(8sr)^2} + \frac{1^2 3^2 5^2 7^2}{\lfloor 4\,(8sr)^4} -$

$$Q = \frac{1^2}{8sr} - \frac{1^2 3^2 5^2}{\lfloor 3\,(8sr)^3} + \dots$$

and $G_0(sr)$ is got by turning sin to cos and cos to − sin in $J_0(sr)$ divergent, or is

$$G_0(sr) = -\frac{2}{\pi}\left\{\left(+\frac{s^2r^2}{2^2} - \left(1+\frac{1}{2}\right)\frac{s^4r^4}{2^24^2} + \dots\right) + J_0(sr)\left[\log\left(\frac{sr}{2}\right) + .5772\right]\right\}$$

so as to be + at origin.

The symmetry is seen in the integrals, & in developments, & it is much the easiest standardization to work practically, I find.

As regards proof of the formidable integrals on last page, I do it by two ways of working out the same physical problem a problem of electric convection.

I dont wonder mathns avoid the second fn. It is so nasty. It gave me a lot of trouble before I got to the bottom of it & then only by ultra-ultra-mathematics. Then again, there are several ways of standardising the K_0 and G_0 fns. Some of them are wretched, making G_0 oscillate about axis in a one sided way!

But are there any tables of the K_0 and G_0 functions?

[The 2nd Edn of your bicycle will not sell, I think.]

Helmholtz's paper. It deserves a lot of explanation as well as expansion. It so happened that I had myself evolved a

inwards both ways, there was scarcely space to pass, and then only
by forcing ones way through the jungle. It was $1\frac{1}{4}$ miles long!
There are many short lanes like that, but such a long one, without
any cross paths, I never saw. 14 | 30

Though only 7 or 8 miles from Torbay, the people here are distinctly
different in their appearance, manners, and names. It is curious
to see a different set of names; I dont mean common names to
to found everywhere, but the local names. But what struck me
forcibly last bankholiday when I went to Exmouth & other places
about there, say 15 miles north, was the distinctly metropolitan
appearance of the places; ~~it~~ was like going through outlying
suburbs of London where old fashioned houses and brick walls
& gardens are to be seen. But the holiday makers at Exmouth
(mostly from Exeter, I expect but some also from Torbay by steamer) were very well behaved, and quite
a contrast to the 'arries and 'arriets who parade on
bankholidays in London, ornamented with paper ribbons, &
rather foulmouthed, though in a good tempered way. What a
pity it is that large places should find it necessary to develop
the worst along with the best, ~~the~~ Never mind moralising — there's
too much of that. But I am sure that climate has a good deal
to do with it; a soft climate & mild manners go together; a
harsh climate and rough brutality; as witness the north east of England
& the S.W. To compensate, they have energy in the N.E., and are
lazy in the S.W. There is a bat now flying about in my
sitting room. Must have got in at the window upstairs, &
then come down. Shall try & catch him; never examined one
at close quarters. He goes round & round, & may smash something
before he has done. Not having any salt at hand, obliged to let him
go. Opened front door, & he soon found the way out. But he went
out too straight for a bat. Perhaps it was the old croaker out for a
holiday. The change of tone is of course more marked at Exeter,

— mouth ?

and if you go on to the town by Portland, (name forgotten) where George III used to live, the metropolitan air is striking. Old fashioned style, of course.

Bottled up wisdom! I am afraid no one has ~~any~~ much in the cathode ray question, without disrespect to your suggestions, which are always ingenious, & sometimes hyperphysical. That's where a doubt lies. We dont know the boundaries of physics, & what seems fundamental may only seem so. It is possible that a very simple theory may be found to explain cathode rays very well; & it may be essentially complicated in theory, as in fact it seems at present when at short intervals some more puzzling facts are turned out.

The man of brass has not redeemed his promise yet about his new submarine cable, nor about his lightship experiments; all in the dark about the facts of the case. Time is running short. I expect he will get his Knighthood at last, this Jubilee. Perhaps the Lodge & Marconi arrangement might suit lightships better than cables. But it is very like heliographic signalling, in some respects.

Yours sincerely

Oliver Heaviside

I hope you will not get drowned in crossing the Atlantic.

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